Department of Physics, IIT-Kanpur

Time : 2 hrs. **PhD Admission Test Dec 2016** **Total Marks: 100**

(1) Figure 1 below shows a triple pendulum. The three masses can swing as illustrated in the right figure. The horizontal line is a thin massless rigid rod and the other lines are taut strings. Find the Lagrangian for the system in terms of the θ-variables (as shown in the figure) as the generalised coordinates, **for small oscillations**. [20]

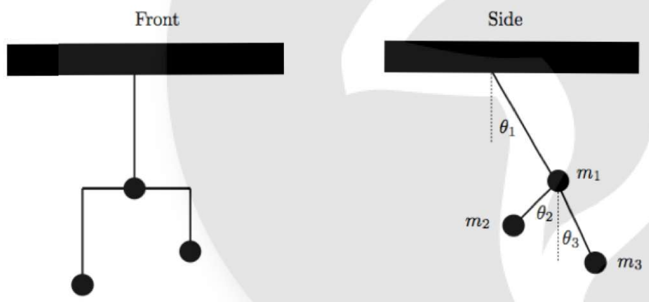


Figure 1:

(2) A beam of spin $\frac{1}{2}$ atoms goes through a series of Stern-Gerlach-type measurements as follows:

- The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.
- The second measurement accept $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms,
  where s_n is the eigenvalue of the operator S_n, the angular momentum operator for spin-$\frac{1}{2}$ particle in the direction of the unit vector $\hat{n}$, making an angle β with respect to the z-axis.
- The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam? [20]

(3a) Show explicitly that if $f(z) = z/\overline{z}$, the limit $lim_{z\to 0}\, f(z)$ does not exist. (Here, $z = x + iy$ and $\overline{z} = x - iy$) [4]

(3b) Let us consider a function of complex variables $f(z) = u(x, y) + iv(x, y)$. Find out $v(x, y)$ and $f(z)$ when $u(x, y) = \sinh x\ \sin y$. (Write down $f(z)$ in terms of z) [4+4]

(3c) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Show that $|\int_C \frac{z+4}{z^3-1}\, dz| \leq \frac{6\pi}{7}$. [5]

(3d) Find the type of singularities (for poles mention the orders and for branch points draw the branch cuts):
(a)$(z^2 - z - 2)^{1/3}$, (b) $-i\log(z + (z^2 - 1)^{1/2})$, (c) $\frac{z^{1/2}-1}{z-1}$, [3+3+3]

(4a) One can calculate all the thermodynamic properties of black body radiation using statistical mechanics through the partition function Z and Helmholtz free energy $F = -k_B T \log Z$. Using the partition function of ideal Bose gas (with zero chemical potential) and the energy dispersion $\epsilon = pc$ (p is momentum and c is the velocity of light), show that the Helmholtz free energy of blackbody radiation at temperature T in a volume V is $F \propto VT^4$. [6]

(4b) In early universe, when the temperature was about $3000K$, the matter and cosmic radiation decoupled and since then they expanded separately. The approximate temperature of the cosmic (black-body) radiation now is $3K$. If the expansion is assumed to be adiabatic, what is the ratio of the volumes of the universe between now and the time of decoupling of the matter and the radiation? [4]

5. (a) Do the following arithmetic operations to the correct number of significant figures:

(i) Find the sum of 831, 1.2, 0.073 and 3.475 [**2**]

(ii) Find the product 0.0062 and π [**2**]

(b) A student weighs water in a beaker and gets the value 20.127 ± 0.005 g. He dissolves some salt in the water and weighs the beaker again and finds the new mass to be 20.183 ± 0.007 g. What is the mass of the dissolved salt with its maximum uncertainty? [**2**]

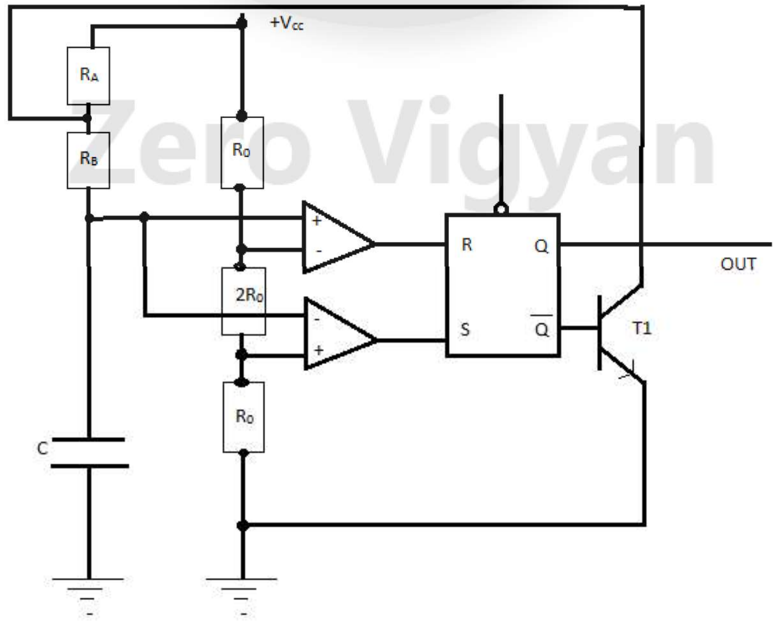
(c) A student wants to determine the spring constant (k) of a spring by suspending mass from it and measuring the period of oscillation of the system. Determine the best value of k with its uncertainty from the sample data given below. Uncertainties in the measurement of mass and period of oscillation are not known. [**4**]

Mass suspended (g)	75	90	100	120	150
Period of oscillation (s)	0.86	0.95	1.00	1.10	1.22

6. (a) Derive the expression for output frequency of the modified 555 square wave generator as shown below. [**3**]

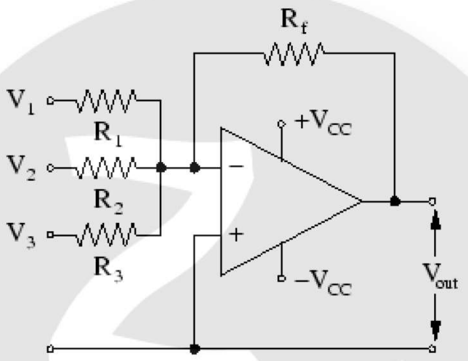
(b) Plot the output waveform at 'Q 'and across the capacitor. [**2**]

Note: Contacts points in the wires are indicated by solid (black) circles.



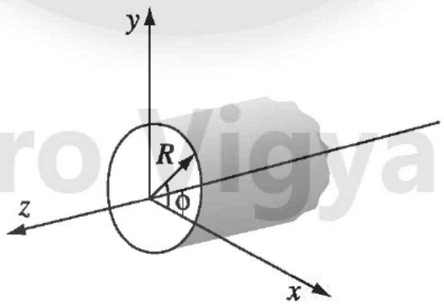
(c) In the circuit shown below $V_{CC} = 12$ V and $R_1 = 1$ kΩ, $R_2 = 2$ kΩ, $R_3 = 4$ kΩ and $R_f = 5$ kΩ. It is given that the opamp output can reach $\pm V_{CC}$.

(i) If $V_1 = 5$ V, $V_2 = -8$ V and $V_3 = -10$ V what is V_{out}? [**2**]
(ii) If $V_1 = 5$ V, $V_2 = 2$ V and $V_3 = 9$ V determine V_{out}. [**3**]



7. (a) The potential on the surface of an infinite cylinder of radius *R* is specified as (please see figure below) :

$$V(R,\phi) = V_o \text{Sin}(3\phi)$$

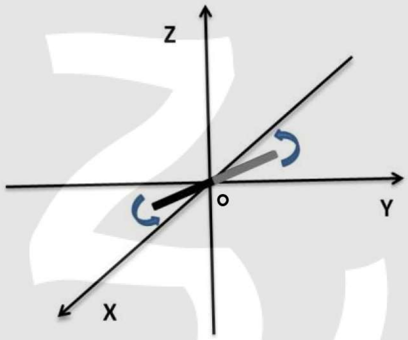


(i) Write down the general form of all the Maxwell's equations (differential forms) in free space. [**3**]
(ii) Write the same Maxwell's equations in the present case for both inside and outside the cylinder. [**2**]
(iii) How can the scalar potential be obtained from the above Maxwell's equations? Find the potential inside V_{in} ($r \leq R$, ϕ) the cylinder (assume general solution as given below, no need to solve the differential equations). Give reasons. [**5**]

(**Given :** the general form of solution of the Laplace's equation in cylindrical polar coordinate is

$$V(r,\emptyset) = a_o + b_o \ln r + \sum_{n=1}^{\infty}[r^n(a_n \cos n\emptyset + b_n \sin n\emptyset) + r^{-n}(c_n \cos n\emptyset + d_n \sin n\emptyset)]$$)

(b) A thin rod of length 2l has a uniform distribution of positive charge +q on one half and negative charge −q on the other half. It lies in the XY plane and rotates at angular frequency ω around the Z-axis passing through the origin o. Answer the following questions.



(i) Express the time-dependent electric dipole moment vector $\vec{P}$ as a complex quantity. [**4**]
(ii) Using the result in (i), find the time-averaged power radiated per unit solid angle, dW/dΩ, far from the dipole. Give the result as a function of the spherical angle θ that describe some direction in space, outward from the dipole. [**6**]

[**Given:** The fields of an oscillating electric dipole are given as follows:

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{P})\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{n}\times\vec{P})\times\hat{n} + [3\hat{n}(\hat{n}.\vec{P}) - \vec{P}]\left(\frac{1}{r^2}-\frac{ik}{r}\right)\right\}\frac{e^{ikr}}{r},$$

where $\vec{P}$ is the dipole moment vector, $\vec{r}$ is the radius vector, $\hat{n}$ is the unit vector along **r**
c is velocity of light and $k = \omega/c$]